

## Linear Positive Machines\*

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### ABSTRACT

The analogy of the Krohn-Rhodes theory of finite state machines is developed for linear positive machines with inputs from a commutative Banach algebra with involution. The notion of prime machine in this context is shown to correspond maximal ideal in the input algebra. Further, every positive machine is shown to lie in the closed convex hull of its primes.

An interesting theory of finite state automata was developed [1] which included an interesting definition of state as well as a complete decomposition theory. Kalman [2] adapted this theory to linear dynamical systems without topology, his main tool being a module-decomposition theorem. We shall consider here a natural extension of these ideas to linear machines with (1) the set of inputs as a commutative Banach algebra with involution, and (2) the outputs as a positive linear functional on the input algebra. We developed a dimension theory for these machines, in such a way that a positive linear machine is one-dimensional if and only if it is an evaluation of the Gelfand transform at a symmetric point of the maximal ideal space. By example we shall show that the condition of positivity is essential in order that multiplicative machines are the prime machines related to dimension.

### DEFINITIONS AND PRELIMINARIES

We shall consider a Banach algebra  $\mathcal{A}$  with involution  $*$  and identity  $e$ .  $\mathcal{A}^*$  will be the dual space of  $\mathcal{A}$ . The following definitions are appropriate:

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DEFINITION 1. A linear machine  $f$  is an element of  $\mathcal{A}^*$ .

DEFINITION 2 (Krohn-Rhodes). The subset of  $\mathcal{A}^*$ ,  $\{foL_u\}_{u \in \mathcal{A}}$  is the state space of the linear machine  $f$ , where  $foL_u(v) = f(uv)$ .

With every linear machine  $f$  we associate the ideal

$$\mathcal{I}_f = \{x \in \mathcal{A} \mid f(ux) = 0 \quad \text{for all} \quad u \in \mathcal{A}\}.$$

THEOREM 1. The state space of a linear machine  $f$  is the ring  $\mathcal{A}/\mathcal{I}_f$ .

*Proof.* Consider the natural map  $u \xrightarrow{\varphi} foL_u$  this is a representation of the ring  $\mathcal{A}$  on the ring  $\{foL_u\}_{u \in \mathcal{A}}$  (i.e.  $(foL_u) \cdot (foL_v) = foL_{uv}$  and  $foL_u + foL_v = foL_{u+v}$ ). Now  $\varphi$  has kernel  $\mathcal{I}_f$  and the result follows.

DEFINITION 3. The dimension of the linear machine  $f$  is the codimension of  $\mathcal{I}_f$  in  $\mathcal{A}$ .

DEFINITION 4. A linear machine  $f$  is positive if and only if  $f$  is a positive linear functional; that is  $f(xx^*) \geq 0$  for all  $x \in \mathcal{A}$ .

We note that if  $f$  is positive  $\mathcal{I}_f = \{x \in \mathcal{A} \mid f(xx^*) = 0\}$ , a simple consequence of the generalized Swartz inequality. Our main theorem is

THEOREM 2. A positive linear machine  $f$  is  $n$  dimensional if and only if there exist  $n$  distinct linear symmetric multiplicative functionals  $M_i$  such that  $f$  is in the convex hull of  $\{M_i\}_{i=1 \dots n}$ .

*Proof.* Suppose  $f = \sum_{i=1}^n C_i M_i$  with  $C_i > 0$ ; then

$$\mathcal{I}_f = \{x \in \mathcal{A} \mid f(xx^*) = 0\} = \{x \in \mathcal{A} \mid M_i(x) = 0, i = 1 \dots n\}$$

and  $f$  is  $n$ -dimensional since  $\mathcal{I}_f$  has codimension  $n$ .

Conversely, if  $f$  is  $n$ -dimensional then, by [3] Theorem 60, there exist  $\varphi_1 \dots \varphi_n$  distinct linear functionals so that

$$\mathcal{I}_f = \{x \in \mathcal{A} \mid \varphi_i(x) = 0 \quad i = 1, \dots, n\}.$$

Now consider for  $z \in \mathcal{A}$   $\varphi_i(zx) = \beta(x)$  as  $\mathcal{I}_f$  is an ideal

$$\mathcal{I}_f \subseteq \{x \in \mathcal{A} \mid \beta(x) = 0\}$$

and in consequence of the general theorem of linear dependence [4] there exist  $\mathcal{A}_{ij}(z)$  so that

$$\varphi_i(zx) = \sum_{j=1}^n \mathcal{A}_{ij}(z) \varphi_j(x)$$

Further since  $f(x^*) = \overline{f(x)}$  and  $f = \sum C_i \varphi_i$  we may assume without loss of generality that  $\varphi_i(x^*) = \varphi_i(x)$ . But  $A(z) = \{A_{ij}(z)\}$  is just a representation of  $\mathcal{A}$  by  $n \times n$  Hermitian matrices so that

$$\begin{aligned} A(z) + A(w) &= A(z + w), \\ A(z) A(v) &= A(v) A(z) = A(vz), \\ A(\alpha z) &= \alpha A(z). \end{aligned}$$

Denote  $e_1, \dots, e_n$  as the left eigenvectors of  $A(z)$  for all  $z \in \mathcal{A}$  then the eigenvectors  $\lambda_i(z)$  of  $A(z)$  satisfy

$$\lambda_i(\alpha z + \beta v) = \alpha \lambda_i(z) + \beta \lambda_i(v), \quad \lambda_i(zv) = \lambda_i(z) \lambda_i(v).$$

Let  $M_i = (e_i, \varphi)$ , where

$$\varphi = \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{pmatrix};$$

then

$$M_i(zx) = M_i(z) M_i(x)$$

with  $e_i$  chosen so that  $M_i(e) = 1$ . But now it is easily seen, since  $\{e_i\}$  span  $C^n$ ,

$$\mathcal{J}_f = \{x \in \mathcal{A} \mid M_i(x) = 0 \quad i = 1, 2, \dots, n\}$$

since the  $\varphi_i$  were assumed distinct.

Since  $N_f = \{x \in \mathcal{A} \mid f(x) = 0\} \supseteq \mathcal{J}_f$ , it follows that

$$f = \sum_{i=1}^n C_i M_i,$$

where  $f(e) = \sum_{i=1}^n C_i$ . Now  $C_i > 0$  and  $M_i(x^*) = \overline{M_i(x)}$  since  $f$  is positive and  $\mathcal{J}_f$  symmetric.

The preceding theorem has the following extension.

**THEOREM 3.** *For a positive linear machine  $f$ ,  $\dim f = \text{cod } \mathcal{J}_f = \text{cardinality of } S_f$  where  $S_f$  is the support of the unique measure  $\mu_f$  associated with  $f$  by Raikov's Theorem (see [5]).*

*Proof.* Since  $f(\cdot) = \int M(\cdot) \mu_f(dM)$ , then

$$\mathcal{J}_f = \{x \in \mathcal{A} \mid M(x) = 0 \quad \text{for all} \quad M \in S_f\},$$

and the result follows.

DEFINITION 5. For two linear machines  $f$  and  $g$  then  $g \mid f$  iff  $\mathcal{I}_f \subseteq \mathcal{I}_g$ .

DEFINITION 6. A linear machine  $f$  is prime iff  $g \mid f$  implies  $g = Cf$  for some complex number  $C$ .

THEOREM 4. A positive linear machine  $f$  is prime in the class of positive linear machines iff  $\mathcal{I}_f$  is a symmetric maximal ideal, or  $\mathcal{I}_f = \{x \in \mathcal{A} \mid M(x) = 0\}$  where  $M(x^*) = M(x)$  and  $M$  is a multiplicative positive linear machine.

*Proof.* Suppose  $\mathcal{I}_f$  is symmetric and maximal and  $g \mid f$  or  $g$  a positive linear machine then  $\mathcal{I}_g \subseteq \mathcal{I}_f$  but maximal property implies  $\mathcal{I}_g = \mathcal{I}_f$  or

$$\mathcal{I}_f = \{x \in \mathcal{A} \mid M(x) = 0\} = \{x \in \mathcal{A} \mid g(xx^*) = 0\} \subseteq \{x \in \mathcal{A} \mid g(x) = 0\}$$

or  $g(x) = \alpha M(x)$  and similarly  $f(x) = \beta M(x)$  or  $g(x) = (\alpha/\beta)f(x)$ .

Suppose  $f$  is prime but  $\mathcal{I}_f$  is not maximal then  $\exists$  a maximal ideal  $\mathcal{I}_M \supseteq \mathcal{I}_f$  where  $\mathcal{I}_M = \{x \in \mathcal{A} \mid M(x) = 0\}$  and  $M$  is symmetric so that  $M \mid f$  and  $M \neq f$  contradiction.

DEFINITION 7.  $P(f) = \{g \in \mathcal{A}^* \mid g \mid f, g \text{ is a prime}\}$ ,

$$P^+(f) = \{g \in \mathcal{A}^* \mid g(xx^*) \geq 0 \quad x \in \mathcal{A}, g \in P(f)\}.$$

Now the Krein-Millman theorem implies

THEOREM 5. Every positive linear machine  $f$  is in the closed convex hull of  $P^+(f)$ .

These results lead to the conjecture that positivity can be removed and Theorem 2 and 4 remain valid when the symmetry assumption is deleted for the  $M_i$ . This is false, for consider  $l'(z^+)$  and  $-1 < x_0 < 1$ ; then

$$f(a) = \sum_{n=0}^{\infty} a_n x_0^n + \sum_{n=1}^{\infty} n a_n x_0^{n-1}$$

is a 2-dimensional linear machine but it is not a finite combination of multiplicative machines! In this case the set of positive linear machines are just the Hausdorff moment sequences for measures on  $[-1, 1]$ . However,

$$f(\mathcal{A}) = \frac{1}{2\pi i} \int_{|z|=1} \frac{\mathcal{A}(z)}{z - x_0} dz + \frac{1}{\pi i} \int_{|z|=1} \frac{\mathcal{A}(z)}{(z - x_0)^2} dz$$

exhibits  $f(\mathcal{A})$  as an average of the multiplicative functionals

$$\mathcal{A}(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{where} \quad \mathcal{A} = \{a_i\}_{i=0}^{\infty}.$$

In fact,

$$\|f(\mathcal{A})\| \leq \|x_0\| \|\mathcal{A}\| + \left(1 + \frac{1}{\log \frac{1}{\|x_0\|}}\right) \|\mathcal{A}\| \quad \text{so that} \quad f \in (l(z^+))^*.$$

In consequence of these results, the dimension theory of linear machines is just the study of the ideal structure of Banach algebras.

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